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MINIMAX THEOREM FOR FUNCTIONS ON THE CARTHESIAN PRODUCT OF BRANCHING POLYLINES

Abstract. The paper proves the minimax theorem for a specific class of functions that are defined on branching polylines in a linear space, not on convex subsets of a linear space. The existence of a saddle point for such functions does not follow directly from the classical minimax theorem and needs individual consideration based both on convex analysis and on graph theory. The paper presents a self-sufficient analysis of the problem. It contains everything that enables plain understanding of the main result and its proof and avoids using concepts outside the scope of obligatory mathematical education of engineers. The paper is adressed to researchers in applied mechanics, engineering and other applied sciences as well as to mathematicians who lecture convex analysis and optimization methods to non-mathematicians.

Keywords: minimax, saddle point, convex analysis, optimization, branching polyline.

The minimax theorem, also known as the saddle point theorem, states one of the fundamental concepts in economics, mechanics, electrical engineering and other applied nonmathematical sciences. Classical version of the minimax theorem has been formulated and proven by J. von Neumann in 1928 [1], generalized by M. Sion in 1958 [2] and newly proven in [3–5].

This paper proves the minimax theorem for specific functions that do not satisfy the conditions of the minimax theorem in its commonly used formulation. These functions are defined on non-convex subsets of linear spaces referred to as branching polylines. The existence of a saddle point for such functions cannot be resolved by a mere reference to the classical minimax theorem and needs individual consideration. The paper presents a self-sufficient analysis of the problem. It contains everything that enables a plain understanding of the main result and its proof and avoids using concepts outside the scope of obligatory mathematical education of engineers.

1. DEFINITIONS AND FORMULATION OF THE MAIN RESULT

Let $\mathbb{R}$, $\mathbb{N}$, $\mathbb{N}^*$ be sets of real numbers, nonnegative and positive integers, respectively. Let $\mathbb{R}^k$, $k \in \mathbb{N}^*$, be a k-dimensional linear space with Euclidean metric $\Delta^k : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, where value $\Delta^k(x, x')$ is the distance between points $x \in \mathbb{R}^k$ and $x' \in \mathbb{R}^k$. A line segment between points $x, x' \in \mathbb{R}^k$ is denoted as

$$[x, x'] = \{\alpha \cdot x + (1-\alpha) \cdot x' \mid 0 \le \alpha \le 1\}.$$

For a given set X and a function $f : X \to \mathbb{R}$ symbols “argmax” and “argmin” are used to express the sets

$$\arg\max_{x \in X} f(x) = \{u \in X \mid f(u) = \max_{x \in X} f(x)\},$$

$$\arg\min_{x \in X} f(x) = \{u \in X \mid f(u) = \min_{x \in X} f(x)\}.$$

Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be closed bounded sets and $f : X \times Y \to \mathbb{R}$ be a continuous function of two arguments. For any such function the inequality

$$\min_{x \in X} \max_{y \in Y} f(x, y) \ge \max_{y \in Y} \min_{x \in X} f(x, y) \tag{1}$$